Übungen zur Vorlesung „ Stochastic Processes “

Abgabetermin: Montag, 30.04.01, in der Vorlesung

1. *Downward excursions on the way up*
Let X be simple random walk on $\mathbb{Z}$. Define the *current maximum* of X as $M_n := \max\{X_k : 0 \le k \le n\}$ and the *current downward excursion depth* as $Y_n := M_n - X_n$. Show that, for $x < z \in \mathbb{Z}$,

$$\mathbf{P}_x\left[\max_{0\le k\le R_z} Y_k < y\right] = \left(\frac{y}{1+y}\right)^{z-x}, \quad y = 1, 2, \dots$$

where R_z denotes the first passage time of X to z.

2. Consider a Markov chain on $\mathbb{N}_0$ with transition probabilities $P(i, i+1) := a_i$, $P(i, 0) := 1 - a_i$, $, i \ge 0$, where $(a_i)_{i\ge 0}$ is a sequence of numbers in $(0, 1)$. Let $b_0 := 1$, $b_i := a_0 a_1 \dots a_{i-1}$ for $i \ge 1$. Show that the chain is
(a) recurrent if and only if b_i converges to zero
(b) positive recurrent if and only if the sum of the b_i is finite.
Find the equilibrium distribution in case (b).

3. (taken from Norris (97), p. 33) The *rooted binary tree* is an infinite graph T with one distinguished vertex r from which comes a single edge; at every other vertex there are three edges and there are no closed loops. The random walk on T jumps from a vertex along each available edge with equal probability. Show the random walk is transient.

4. (a) Let (X_n) be a Markov chain on a discrete state space S_0, and let $V(y)$ denote the number of visits of the path X in y. Show that

$$\mathbf{E}_x[V(y)] \le \mathbf{E}_y[V(y)], \quad x, y \in S_0$$

b) Let P be a stochastic matrix on S_0, and put $G(x, y) := \sum_{n=0}^{\infty} P^n(x, y)$. Show that

$$G(x, y) \le G(y, y), \quad x, y \in S_0.$$

5. * (following Lawler(95)) Let $Z_1, Z_2, \dots$ be i.i.d. $\mathbb{Z}$-valued random variables with expectation 0, and $X_n := x + Z_1 + \dots + Z_n, \quad n = 1, 2, \dots$ be the random walk built from the increments Z_j.
Let $G_n(0, y) := \mathbf{E}_0[\sum_{j=0}^{n} I_{X_j = y}]$ be the expected number of visits to y in the first n steps.

(a) Recall that the law of large numbers implies that for each $\varepsilon > 0$

$$\lim_{j\to\infty} \mathbf{P}_0[| X_j |> j\varepsilon] = 0.$$

Show that this implies that for every $\epsilon > 0$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{|y|>\varepsilon n} G_n(0, y) = 0$$

and consequently

$$\lim_{n\to\infty} \frac{1}{n} \sum_{|y|\le\varepsilon n} G_n(0, y) = 1$$

(b) Using (a) and Exercise 4, show that for each $M < \infty$ there is an n such that $G_n(0, 0) \ge M$, and conclude that (X_n) is a recurrent Markov chain.