„ Stochastic Processes "
Assignment 2

Due: Monday, May 7, 2001, in class

6. Show that a stochastic matrix P on a finite state space has at least one equilibrium distribution.

7. What is the expected number of visits of an ordinary random walker on $\mathbb{Z}$ to $z = 9999$ before the first return to the starting point when starting out from 0?

8. For fixed $N \in \mathbb{N}$, put $\Omega := \{\eta = (\eta^{(1)}, \dots, \eta^{(N)}) : \eta^{(i)} \in \{ \text{ plus, minus } \}\}$, and for $\eta \in \Omega$, let $f(\eta)$ be the number of plus components in η. Consider the following stochastic dynamics on Ω: Given η, pick an i uniformly from $\{1, \dots, N\}$ and flip the sign of $\eta^{(i)}$.

a) Let (η_n) be a Markov chain following this dynamics. Show that $X_n = f(\eta_n)$ is a Markov chain on $S_0 := \{0, \dots, N\}$, compute its transition probability P and its equilibrium distribution. (Hint: First look for an equilibrium distribution of (η_n))

b) Does $P^n(z, .)$ converge as $n \to \infty$? Does $P^{2n}(z, .)$ converge as $n \to \infty$? And if so, how does the latter limit relate to the equilibrium distribution of P?

c) Consider the following modification of the P-dynamics: In each step, one performs with probability 1/2 a step following P, and with probability 1/2 one takes a rest. This leads to the the transition matrix $Q := (1/2)(P + I)$, where I is the unit matrix. Does $Q^n(z, .)$ converge, and if so, to which limit?

9. A random variable $T = T(X)$ taking its values in $\{0, 1, 2, \dots\} \cup \{\infty\}$ is called a *stopping time* if, for all $n = 0, 1, 2, \dots$, the event $\{T = n\}$ depends only on $X_0, \dots, X_n$, i.e. $\{T = n\} = \{(X_0, \dots, X_n) \in C_n\}$ for some $C_n \in S_0^{\{0,\dots,n\}}$. Which of the following random times are stopping times:

a) the first passage time to z

b) the second passage time to z

c) the last passage time to z?

10.* Let (Z_n) be a coin tossing sequence with $\mathbb{P}[Z_n = H] =: p$, $\mathbb{P}[Z_n = T] =: q$, and for given ℓ, let $X_n = (Z_{n-\ell+1}, \dots, Z_n)$ be the *pattern of length* ℓ occurring at time n.
a) Is $(X_n)_{n \geq \ell-1}$ a sequence of independent random variables? Is it a Markov chain?
b) What is the expected waiting time w between successive occurences of the pattern $HTHT$?
c) Show that w equals the expected waiting time from the occurrence of HT to that of $HTHT$.
d) Shtow that the expected waiting time from time 0 to the first occurrence of HT equals $\frac{1}{pq}$.
e) Combine c) and d) to show that the expected waiting time from time 0 to the first occurrence of $HTHT$ equals $\frac{1}{pq} + w$.