# „Stochastic Processes"
# Assignment 3

Due: Monday, May 14, 2001, in class

**11.** For a stopping time $T = T(X)$, we say that the event $A$ is determined by $(X_0, \dots, X_T)$ if for all $n \geq 0$,

$$A \cap \{T \leq n\} = \{(X_0, \dots, X_n) \in B_n\}$$

for some $B_n \subseteq S_0^{\{0,\dots,n\}}$.
Let $X$ be a Markov chain on $S_0$, $T$ be a stopping time and $A$ be an event determined by $(X_0, \dots, X_T)$. Prove and interprete the following equality:

$$\begin{aligned} &\mathbf{P}_\mu[\{(X_T, X_{T+1}, \dots) \in B\} \cap A \cap \{T < \infty\} \cap \{X_T = z\}] \\ = \; &\mathbf{P}_\mu[A \cap \{T < \infty\} \cap \{X_T = z\}] \cdot \mathbf{P}_z[X \in B]. \end{aligned}$$

**12.** Let $S_0$ be a finite set, and $F$ be a random mapping from $S_0$ to $S_0$. For $x, y \in S_0$ put $P(x, y) := \mathbf{P}[F(x) = y]$. Moreover, let $F_1, F_2, \dots$ be a sequence of independent random mappings from $S_0$ to $S_0$, all having the same distribution as $F$. Show

**a)** If $Y$ is an $S_0$-valued random variable independent of $F$ , whose distribution is an equilibrium distribution of $P$, then $F(Y)$ has the same distribution as $Y$.

**b)** Assume that the random subsets $R_n$ of $S_0$ defined by

$$R_n := F_1(F_2(\dots(F_n(S_0))\dots)), \quad n = 1, 2, \dots$$

have an intersection $R := \bigcap R_n$ which $\mathbf{P}$-a.s. consists of only one element, i.e. $R = \{Z\}\mathbf{P}$-a.s. for an $S_0$- valued random variable $Z$. Then $P$ has exactly one equilibrium distribution, and this is the distribution of $Z$.
(Hint: Consider the sequence $Y_n := F_1(F_2(\dots(F_n(Y))\dots))$, $n = 1, 2, \dots$, where $Y$ is as in a))

**13.** Consider a simple random walk ($p = 1/2$) with absorbing boundaries on $\{0, 1, 2, \dots 10\}$. Suppose the following payoff function is given

$$[0, 2, 4, 3, 10, 0, 6, 4, 3, 3, 0].$$

Find the optimal stopping rule and give the expected payoff starting at each site.

**14.** Consider a simple "Wheel of Fortune" game. A wheel is divided into 12 equal-sized wedges. Eleven of the wedges are marked with the numbers $100, 200, \dots, 1100$ denoting an amount of money won if the wheel lands on those numbers. The twelfth wedge is marked "bankrupt". A player can spin as many times as he or she wants. Each time the wheel lands on a numbered wedge, the player receives that much money which is added to his/her previous winnings. However, if the wheel ever lands on the "bankrupt" wedge, the player loses all of his/her money that has been won up to that point. The player may quit at any time, and take all the money he or she has won (assuming the "bankrupt" wedge has not come up). Assuming that the goal is to maximize one's expected winnings in this game, devise an optimal strategy for playing this game and compute one's expected winnings. You may wish to try a computer simulation first.
(From: G.F. Lawler, Introduction to Stochastic Processes, Chapman & Hall, 1995)

**15.** Let $P$ be a stochastic matrix on $S_0$; $f, g$ be nonnegative functions on $S_0, \alpha \in (0, 1]$. Consider the following analytic problems:
Find the smallest $v \geq f$ with
(i) $v(x) \geq Pv(x) - g(x)$
(ii) $v(x) \geq \alpha \cdot Pv(x)$
(iii) $v(x) \geq \alpha \cdot Pv(x) - g(x)$,
for all $x \in S_0$.
Which kind of stopping problems would you associate with (i), (ii), and (iii), respectively ?