Stochastic Processes
Assignment 4

Due: Monday, May 21, 2001, in class

16. *Time reversal.* Let P be an irreducible transition matrix on S_0, having an equilibrium distribution π.

a) Show that

$$Q(x,y) := \frac{1}{\pi(x)} P(y,x)\pi(y), \quad x, y \in S_0$$

is a transition matrix.

b) Compute $\mathbf{P}_\pi[X_0 = y | X_1 = x]$ and $\mathbf{P}_\pi[X_0 = z | X_1 = y, X_2 = z]$.

c) Show that

$$\mathbf{P}_\pi[X_n, X_{n-1}, \dots, X_0) = (x_n, \dots, x_0)] = \pi(x_n) Q(x_n, x_{n-1}) \dots Q(x_1, x_0).$$

Congratulations! You have just shown that the time reversal of a stationary Markov chain is again a stationary Markov chain.

17. *Time reversal of the residual lifetime process.* Consider a renewal chain with life time distribution ϱ on $\mathbb{N}$ and finite expected life time $\mathbf{E}R$. Let ν be the equilibrium distribution of the residual lifetime process (Y_n), and $P(x,y)$ be its transition matrix. (Recall that $\nu(k) = \frac{1}{\mathbf{E}R}\mathbf{P}[R > k]$.)

a) Compute the "dual transition matrix" Q in the sense of Exercise 16a).

b) Compute the *hazard function*: the conditional probability that the lifetime R equals $k+1$, given that it exceeds k.

Congratulations! You have just seen that the time reversal of the staionary residual lifetime process is the stationary age process.

18. We consider three lifetime distributions $\varrho_1, \varrho_2, \varrho_3$:

$$\begin{aligned} \varrho_1(dr) &:= \frac{1}{10} 1_{[0,10]}(r) dr \\ \varrho_2(dr) &:= \delta_{10}(dr) \\ \varrho_3(dr) &:= 1_{[1,\infty)}(r) \frac{1}{2r^3} dr \end{aligned}$$

a) For all three cases, plot the equilibrium density of the residual lifetime process. In which cases does convergence to equilibrium hold ?

b) Compute for all three cases

i) the expected lifetime,

ii) the expected residual lifetime in equilibrium,

iii) the expected size-biased lifetime.

19. Recall that the Gamma distribution with form parameter $k \in \mathbb{N}$ has density

$$g_k(x) := \frac{1}{(k-1)!} x^{k-1} e^{-x}, \quad x \geq 0$$

a) Show that Gamma(k) is the distribution of the sum of k independent, exponential(1)-distributed random variables (Hint: induction).

b) Compute the l-th moment $\mathbf{E}Z^l$ of a Gamma(k)-distributed random variable Z.

c) Which distribution does one obtain by size-biasing a Gamma(k)-distribution?

d) Can you explain the result of c) in the light of a stationary renewal process with lifetime distribution Gamma(k)?

By the way, there even holds a more general fact (which can also be nicely interpreted in the light of stationary renewal processes):
Let $Y_1, Y_2, \dots Y_k$ be i.i.d. copies of a random variable Y. Then the size-biased distribution of $Y_1 + \dots + Y_k$ arises as the distribution of $\hat{Y} + Y_2 + \dots + Y_k$, where the size-biased $\hat{Y}$ is independent of $Y_2, \dots, Y_k$.

20. (From: S. Ross, Stochastic processes, Wiley 1996, p. 154)
Consider a single-server bank in which potential customers arrive at a Poisson rate α but only enter when the server is free. Let G denote the distribution of the service time.
a) At what rate do customers enter the bank?
b) What fraction of potential customers enter the bank?
c) What fraction of time is the server busy?