PROF. DR. A. WAKOLBINGER SS 2001

Stochastic Processes
Assignment 5

Due: Monday, May 28, 2001, in class

21. *The Gamma Distribution*
The Gamma(k)-distribution has density:

$$g_k(y) = \frac{1}{\Gamma(k)} y^{k-1} e^{-y}, \quad y > 0$$

(Recall that $\Gamma(k) = \int_0^\infty x^{k-1} e^{-x} dx$)
a) *Laplace transform:*
Let Y be Gamma(k)-distributed, $k > 0$. Show that

$$\mathbf{E}\, e^{-\beta Y} = (1+\beta)^{-k}, \quad \beta > 0.$$

b) Let $Y_i \quad (i = 1, 2)$ be Gamma(k_i)-distributed and independent. What is the distribution of $Y_1 + Y_2$?
(Hint: The distribution of an $\mathbb{R}_+$-valued random variable is uniquely determined by its Laplace transform.)

22. *Poissonian moments*
Let N be Poisson(α)-distributed:

$$\mathbf{P}[N = n] = e^{-\alpha} \frac{\alpha^n}{n!}, \quad n = 0, 1, \dots$$

a) *Laplace transform:* Show that

$$\mathbf{E}\, e^{-\beta N} = \exp(-(1 - e^{-\beta})\alpha)\,.$$

b) Use a) to compute the expectation and variance of N.

23. *Poisson decimated*
Let N be Poisson(α)-distributed and let $X_1, X_2, \dots$ be a coin-tossing sequence with success probability p, independent of N. Let $S_n := \sum_{i=1}^{n} X_i$.

a) Compute $\mathbf{E}\, \exp(-\beta S_n), \quad \beta > 0$

b) Compute $\mathbf{E}\, \exp(-\beta S_N), \quad \beta > 0.$

(Hint: $\mathbf{E} f(S_N) = \sum_{n=0}^{\infty} P(N = n) E f(S_n)$)

c) So how is S_N distributed ?

d) How could you have obtained the result of c) without computation by thinning a Poisson process ?

24. *Poisson with the jitters*
Let $\{X_n\}$ be the points of a homogeneous Poissson process on $\mathbb{R}$ and let $Y_n = X_n + J_n$, where $\{J_n\}$ are i.i.d. random variables independent of $\{X_n\}$. Could it be that $\{Y_n\}$ is also a homogeneous Poissson process? (Check the postulates.)

25. *Queuing*
In the single server bank of Exercise 20 customers arrive at time points forming a homogenous Poisson process with rate α. Assume now that customers enter the bank even when the server is not free, and queue up. Let G be the distribution of the service times.

a) Compute (once again) the expected number of customers arriving while one customer is served.

b) Assume that initially there are no customers waiting. Let Z_n be the number of people in the queue immediately after the n-th customer has been served. Then during the first busy period, i.e. until he server is free again, Z_n has the form $Z_n = 1 + X_1 + \ldots + X_n$, where the X_i are i.i.d. random variables. What is X_i? Assume that the service rate $\frac{1}{\mu}$ equals the arrival rate α. Can it happen that the line becomes so long that the server is never free again ?