A. WAKOLBINGER SS 2001

Stochastic Processes
Assignment 6

Due: Monday, June 11, 2001, in class

26. ***The Yule Process***
Consider a population starting from one ancestor at time 0. Assume that each individual, independently of everything else, gives birth to another individual at rate $\alpha > 0$. (The birth time points of the ancestor's children thus form a homogeneous Poisson process with rate α, but remember that each of these children then also gives rise to its own children – draw a tree of lines of descent.)
Let Z_t be the population size at time t, and T_n be the time when the population size jumps from n to $n+1$. (By convention we put $Z_{T_n} = n+1$.)

a) What are the jump rates of (Z_t) ?

b) Why does T_n have the same distribution as $W_1 + \ldots + W_n$, where the W_j are independent and $\mathrm{Exp}(j\alpha)$-distributed?

c) Why does T_n have the same distribution as $\max_{1\le i\le n} Y_i$, where the Y_i are independent and $\mathrm{Exp}(\alpha)$-distributed?

d) Verify that

$$\mathbf{P}[Z_t > n] = \mathbf{P}[T_n < t] = (1 - e^{-\alpha t})^n.$$

(You've just proved that the size of a Yule process at time t, starting from one ancestor at time 0, has a geometric distribution.)

27. ***Yule processes and Pólya urns***
Recall Pólya's urn: Initially, there are k_1 red and k_2 blue balls in the urn. Each time a ball is drawn at random, and then replaced together with a new ball of the same colour.
Let $(Z_t^{(1)}, (Z_t^{(2)})$ be two Yule processes, each with parameter 1, starting from k_1 and k_2 ancestors at time 0. (So apart from the initial condition, $Z^{(i)}$ is as in the previous exercise.)
Denote by J_n the time when the total population size $Z^{(1)} + Z^{(2)}$ jumps from $k_1 + k_2 + n - 1$ to $k_1 + k_2 + n$.

a) Why does $Z^{(1)}_{J_n}$ have the same distribution as the number of red balls in a Pólya urn (with initially k_1 red and k_2 blue balls) after the n-th drawing and replacement.

b) Why is the asymptotic proportion of red balls in a Pólya urn (with initially k_1 red and k_2 blue balls) distributed like $\frac{X_1}{X_1+X_2}$, where the X_i are independent and $\mathrm{Gamma}(k_i)$-distributed?
(Hint: Combine a) and the result of Exercise 26)

28. ***Discrete time poissonized***
Let S_0 be a finite or countable state space, Π be a stochastic matrix on S_0, and $\alpha > 0$. Consider a Markov chain (X_t) in continuous time with transition probability

$$\mathbf{P}_x[X_t = y] := \sum_{n=0}^{\infty} e^{-\alpha t} \frac{(\alpha t)^n}{n!} \Pi^n(x, y).$$

Describe the stochastic dynamics of X_t. What is its Q-matrix?
How does a homogeneous Poisson counting process (N_t) fit into this picture? What is its Q-matrix?

29. ***The Coalescent***
Consider a population of initially k individuals. Think of each of the $\binom{k}{2}$ pairs being equipped with a timer which rings at an Exp(1)-distributed time (all these times being independent). When the first timer rings, the corresponding pair coalesces into one individual, so that there are $k-1$ individuals left. Then the same game is played again, now starting from $k-1$ individuals, and so on.

a) What is the expected time of coalescence from k individuals to one individual?

b) Construct a random path of an $\mathbb{N}$–valued Markov chain with Q-matrix

$$Q(n, n-1) = -Q(n, n) = \binom{n}{2}$$

"entering from infinity" at time $t = 0$.