Stochastic Processes
Assignment 7

Due: Monday, June 18, 2001, in class

30. *Rescaling a random walk to get a diffusion*
Consider a continuous time random walk on $\frac{\mathbb{Z}}{\sqrt{n}}$ with jump rates $q_x^{(n)} := n$, and jump matrix $J^{(n)}(x, x \pm \frac{1}{\sqrt{n}}) = \frac{1}{2}$. Let $Q^{(n)}$ be the corresponding Q-matrix and write

$$(Q^{(n)}f)(x) := \sum_{z \in \frac{1}{\sqrt{n}} \cdot \mathbb{Z}} Q^{(n)}(x,z) f(z)$$

a) If f is twice continuously differentiable, and $x_n \to x$, show that

$$\lim_n (Q^{(n)}f)(x_n) = \frac{1}{2} f''(x).$$

b) How should $q^{(n)}$ and $J^{(n)}$ be modified so that

$$\lim_n (Q^{(n)}f)(x_n) = b(x)f'(x) + \frac{1}{2}\sigma^2(x) f''(x)$$

for given functions b and σ ?

31.a) ***Critical Galton-Watson process***
Consider a population in which each individual, independently of all the others, gives birth to a new individual at rate $\frac{1}{2}$, and dies at rate $\frac{1}{2}$.
Argue that the Q-matrix of the population size process X satisfies

$$Q(k, k+1) = Q(k, k-1) = -\frac{1}{2} Q(k,k) = \frac{1}{2} k.$$

Compute $(Qf)(k)$.
Do you see a connection with the diffferential operator

$$\frac{\sigma^2 x}{2} \frac{d^2}{dx^2} ?$$

b) ***Subcritical Galton-Watson process with immigration***
Modify the dynamics of a) as follows: Each individual gives rise to a newborn at rate $\frac{1}{2}$, and dies at rate $\frac{1}{2} + \mu, \quad \mu > 0$. In addition, immigrant individuals appear (out of nowhere) at rate $\mu \cdot c$. Once in the population, these individuals behave like all the others.
Compute $(Qf)(k)$, where Q is the Q-matix of the population size process.
Can you think of a way of scaling space and time such that the differential operator

$$\frac{\sigma^2 x}{2} \frac{d^2}{dx^2} - \mu(x-c)\frac{d}{dx}$$

appears in the limit? (Here, σ is a positive constant.)

32. *Probability of extinction*
Let X_t be the birth-and-death process with birth rate $\beta_n = n\beta$ and death rate $d_n = n\delta$, i.e. X_t is an $\mathbb{N}_0$-valued Markov chain with Q-matrix

$$Q(n, n+1) = n\beta, \quad Q(n, n-1) = n\delta, \quad Q(n,n) = -n(\beta + \delta).$$

a) Interpret (X_t) as population size process of a Galton Watson process (see Excercise 31)

b) Show that $\mathbf{P}_2[X_t = 0] = \mathbf{P}_1[X_t = 0]^2$.

c) Establish the backward equation for

$$u(t,n) := \mathbf{P}_n[X_t = 0].$$

From this, obtain a differential equation for

$$\phi(t) := \mathbf{P}_1[X_t = 0]$$

and compute $\phi(t)$.

33. *from S. Ross, Stochastic processes, 2nd ed., Wiley 1996*
a) For a pure birth process with birth parameters $\beta_n, n \geq 0$, compute the mean, variance, and Laplace transform of the time it takes the population to go from size 0 to size N.

b) Consider a birth and death process with birth rates $\{\beta_n\}$ and death rates $\{\mu_n\}$. Starting in state i, find the probability that the first k events be all births.