Stochastic Processes
Assignment 8

Due: Monday, June 25, 2001, in class

34 a) Let S_0 be a finite state space, P be a stochastic matrix and π be a probability distribution on S_0.
Show that π is an equilibrium distribution for P provided it fulfills the detailed balance equation

$$\pi(x)P(x,y) = \pi(y)P(y,x) \qquad (x, y \in S_0) \tag{1}$$

(i.e. if for the $(P;\pi)$-chain the flow from x to y is the same as the flow from y to x).

b) Let π be a probability distribution on S_0, and R be a stochastic matrix on S_0. We want to modify R into a P such that the detailed balance equation (1) holds. To this purpose, let us fix $x \in S_0$ and let Z have distribution $R(x,.)$.
Now modify Z as follows. If $Z = y$, and $\pi(x)R(x,y) \le \pi(y)R(y,x)$, then accept the proposed outcome y, in putting

$$Y := y.$$

If $\pi(x)R(x,y) > \pi(y)R(y,x)$, then accept y only with probability

$$p := \frac{\pi(y)R(y,x)}{\pi(x)R(x,y)},$$

i.e. put

$$Y = \begin{cases} y & \text{with probabilty } p, \\ x & \text{with probabilty } 1-p. \end{cases}$$

Let $P(x,.)$ be the distribution of Y. Show that π and P satisfy the detailed balance equation (1).

c) (Hastings algorithm) Let R be an irreducible stochastic matrix on S_0, and π be a probability distribution on S_0 with strictly positive weights whose ratios $r(x,y) := \frac{\pi(x)}{\pi(y)}$ are known for all x, y with $R(y,x) > 0$. Specify the dynamics of an irreducible Markov chain (X_n) with equilibrium distribution π in terms of R and r.

35 Let Z, Z_1, Z_2 be real-valued random variables, and X be a random variable taking its values in some measurable space $(S, \mathcal{S})$.

a) Recall that also constants can be viewed as random variables. What is the conditional expectation of Z, given a constant?

b) Prove the linearity of the conditional expectation:

$$\mathbf{E}[\alpha Z_1 + \beta Z_2 | X] = \alpha \mathbf{E}[Z_1|X] + \beta \mathbf{E}[Z_2|X] \quad \text{a.s.}$$

c) Prove the monotonicity of the conditional expectation:

$$Z_1 \le Z_2 \quad \text{a.s.} \implies \quad \mathbf{E}[Z_1|X] \le \mathbf{E}[Z_2|X] \quad \text{a.s.}$$

d) A transition probability $\Pi(x, dz)$ from S to $\mathbb{R}$ is called *conditional distribution* of Z given $X = x$ if the joint distribution of X and Z is of the form

$$\mathbf{P}[(X,Z) \in (dx,\, dz)] = \mathbf{P}[X \in dx]\,\Pi(x, dz).$$

Write $\mathbf{E}[Z|X]$ in terms of Π.

36 a) Recall that the correlation of two random variables Y_1 and Y_2 with positive variances σ_1^2 and σ_2^2 is given by

$$\kappa = \frac{\mathrm{Cov}[Y_1, Y_2]}{\sigma_1 \sigma_2}.$$

Show that $\frac{\sigma_2 \kappa}{\sigma_1} Y_1$ and $Y_2 - \frac{\sigma_2 \kappa}{\sigma_1} Y_1$ are uncorrelated (i.e. have correlation zero).
(Hint: You may assume without loss of generality (why?) that Y_1 and Y_2 have expectation 0.)

b) Let Z_1 and Z_2 be real-valued random variables with a joint normal distribution, expectations μ_1 and μ_2, variances σ_1^2 and σ_2^2, and correlation κ. Using the fact that random variables with a joint normal distribution are independent iff they are uncorrelated, show that the conditonal distribution of Z_2 given $Z_1 = z_1$ is normal with mean $\mu_2 + \frac{\sigma_2 \kappa}{\sigma_1}(z_1 - \mu_1)$ and variance $\sigma_2^2(1 - \kappa^2)$.