Stochastic Processes
Assignment 9

Due: Monday, July 2, 2001, in class

37 Compute the conditional expectation of the smallest of five independent, uniformly on $[0,1]$ distributed random variables, given that the biggest of them takes the value $2/3$.

38 Find an example

a) of a process (X_n) with $\mathbf{E}X_n \equiv 0$, which is no martingale,

b) of a martingale (X_n) with $\mathbf{E}X_n \equiv 0$ and $X_n \to -\infty$ a.s.

Hint to part b): Think of a “fair game” with smaller and smaller success probabilities.

39 Let $\mathbb{F}$ be a filtration and $X = (X_n)$ be $\mathbb{F}$-adapted, all the X_n being integrable. Put

$$D_n := \mathbf{E}[X_n - X_{n-1} | \mathbb{F}_{n-1}]; \; B_0 := 0, \quad B_n := \sum_{i=1}^{n} D_i, \quad n = 1, 2, \ldots$$

Show:

i) $M_n := X_n - B_n, \quad n = 0, 1, \ldots,$ is an $\mathbb{F}$-martingale.

ii) $(B_n)_{n \geq 1}$ is $\mathbb{F}$-previsible.

iii) The decomposition $X = M + B$ into a martingale M and a previsible process B with $B_0 = 0$ is a.s. unique. (It is called **Doob–decomposition** of X.)

40 Consider Pólya’s urn, with one red and one blue ball initially (recall Exercise 27). Let X_n be the number of red balls and $M_n := \frac{X_n+1}{n+2}$ their proportion immediately after the n-th drawing.
Does M_n converge a.s.? If so, what is the distribution of the limiting random variable?