Übungen zur Vorlesung „Stochastic Processes“

Abgabetermin: Dienstag, 23.04.02, in der Vorlesung

1. Let X be simple random walk on $\mathbb{Z}$.
a) Put $g(x) := \mathbf{P}_x[X_n = 0 \text{ for some } n \geq 0] = \mathbf{P}_x[V_0 > 0]$. Argue that $g(x) = 1/2(g(x-1) + g(x+1))$. Determine $g(0)$. Can it be that $g(1) < 1$? Conclude from your result that simple random walk on $\mathbb{Z}$ recurrent.
b) Fix $y > 0$ and compute $h(x) := \mathbf{P}_x[X \text{ hits } y \text{ before it hits } 0]$, $0 \leq x \leq y$.

2. *Downward excursions on the way up*
Let X be simple random walk on $\mathbb{Z}$.
a) Decompose X (starting in 0) on its way up to ∞ into the "downward excursions" $D^{(\ell)}$ between the random times when X hits level ℓ for the first time and when X hits level $\ell + 1$ for the first time, $\ell = 1, 2, \ldots$. Show, using 1 b), that the probabilities $\mathbb{P}[D^{(\ell)} \text{ hits } 0]$ sum to ∞.
b) The second Borel-Cantelli-Lemma says that in coin-tossing with success probabilities $p_1, p_2, \ldots$ summing to infinity, infinitely many successes occur with probability 1. How does this relate to a) and the recurrence of X?
c) Define the *current maximum* of X as $M_n := \max\{X_k : 0 \leq k \leq n\}$ and the *current downward excursion depth* as $Y_n := M_n - X_n$. Show that, for $x < z \in \mathbb{Z}$,

$$\mathbf{P}_x\left[\max_{0 \leq k \leq R_z} Y_k < y\right] = \left(\frac{y}{1+y}\right)^{z-x}, \quad y = 1, 2, \ldots$$

where R_z denotes the first passage time of X to z.

3. (taken from Norris (1997), p. 33) The *rooted binary tree* is an infinite graph T with one distinguished vertex r from which comes a single edge; at every other vertex there are three edges and there are no closed loops. The random walk on T jumps from a vertex along each available edge with equal probability. Show the random walk is transient.

4) *The idea of the exercise is that you should write a little program in your favorite language, and experiment rather than argue theoretically. But if you like, you may also wait for theoretical arguments to come in the lecture.*
Consider the Markovian dynamics on $\{1, 2, 3, 4, 5\}$ specified in the picture (where $a \geq 2$).
a) First consider the case $a = 2$, i.e. no rest in state 5. Which part of the time is spent by X in the states in the long run? Does the initial state play a role in this question?
b) How should one choose a such that the relative time spent in state 5 equals in the long run approximately the relative time spent in $\{1, 2, 3, 4\}$?