Übungen zur Vorlesung „ Stochastic Processes “

Abgabetermin: Dienstag, 30.04.02, in der Vorlesung

5. a) Let z and y be two different states in the same recurrent class. Recall that $m_z(y)$ was defined to be the expected number of visits in y in an excursion from z. Show that

$$m_z(y) = \frac{\mathbf{P}_z[R_y < R_z]}{\mathbf{P}_y[R_z < R_y]} < \infty$$

Hint: Argue first that

$$m_z(y) = \mathbf{P}_z[R_y < R_z]\,\mathbf{E}_y\left[\sum_{n=0}^{R_z} I_{\{X_n=y\}}\right],$$

and then that $\sum_{n=0}^{R_z} I_{\{X_n=y\}}$ has under $\mathbf{P}_y$ a geometric distribution with parameter $\mathbf{P}_y[R_z < R_y]$.
b) Assume that S_0 is finite and $z \in S_0$ is recurrent. Conclude from a) that m_z has finite total mass. (Recall from the course that then z is positive recurrent, and $m_z/m_z(S_0)$ is an equilibrium distribution.)
c) Assume that S_0 is finite. Show that there must exist at least one recurrent state.
d) Conclude from c) and b) that every transition matrix P on a *finite* set of states S_0 has at least one equilibrium distribution.

6. What is the expected number of visits of an ordinary random walker on $\mathbb{Z}$ to $z = 2002$ before the first return to the starting point when starting out from 0?
(Hint: Recall the connection between that sort of question and invariant measures!)

7. For fixed $N \in \mathbb{N}$, put $\Omega := \{\eta = (\eta^{(1)}, \dots, \eta^{(N)}) : \eta^{(i)} \in \{ \text{plus}, \text{minus} \}\}$, and for $\eta \in \Omega$, let $f(\eta)$ be the number of plus components in η. Consider the following stochastic dynamics on Ω: Given η, pick an i uniformly from $\{1, \dots, N\}$ and flip the sign of $\eta^{(i)}$.

a) Let (η_n) be a Markov chain following this dynamics. Show that $X_n = f(\eta_n)$ is a Markov chain on $S_0 := \{0, \dots, N\}$, compute its transition probability P and its equilibrium distribution. (Hint: First look for an equilibrium distribution of (η_n).)

b) Does $P^n(z,.)$ converge as $n \to \infty$? Does $P^{2n}(z,.)$ converge as $n \to \infty$? And if so, how does the latter limit relate to the equilibrium distribution of P?

c) Consider the following modification of the P-dynamics: In each step, one performs with probability 1/2 a step following P, and with probability 1/2 one takes a rest. This leads to the the transition matrix $Q := (1/2)(P + I)$, where I is the unit matrix. Does $Q^n(z,.)$ converge, and if so, to which limit?

8. *You can either do this exercise with pencil and paper, or verify the statements empirically for a couple of values of p and q by writing a simulation program. You can achieve full points in either way, and both should be fun.*

Let (Z_n) be a coin tossing sequence with $\mathbf{P}[Z_n = H] =: p$, $\mathbf{P}[Z_n = T] =: q$, and for given ℓ, let $X_n = (Z_{n-\ell+1}, \dots, Z_n)$ be the *pattern of length* ℓ occurring at time n.
a) Is $(X_n)_{n \geq \ell-1}$ a sequence of independent random variables? Is it a Markov chain?
b) What is the expected waiting time w between successive occurences of the pattern $HTHT$?
c) Show that w equals the expected waiting time from the occurrence of HT to that of $HTHT$.
d) Show that the expected waiting time from time 0 to the first occurrence of HT equals $\frac{1}{pq}$.
e) Combine c) and d) to show that the expected waiting time from time 0 to the first occurrence of $HTHT$ equals $\frac{1}{pq} + w$.