Übungen zur Vorlesung „Stochastic Processes“

Abgabetermin: Dienstag, 07.05.02, in der Vorlesung

9. *Time reversal.* Let P be an irreducible transition matrix on S_0, having an equilibrium distribution π.

a) Show that

$$Q(x,y) := \frac{1}{\pi(x)} P(y,x)\pi(y), \quad x,\, y \in S_0$$

is a transition matrix.

b) Compute $\mathbf{P}_\pi[X_0 = y | X_1 = x]$ and $\mathbf{P}_\pi[X_0 = z | X_1 = y,\, X_2 = z]$.

c) Show that

$$\mathbf{P}_\pi[X_n, X_{n-1}, \dots, X_0) = (x_n, \dots, x_0)] = \pi(x_n) Q(x_n, x_{n-1}) \dots Q(x_1, x_0).$$

Congratulations! You have just shown that the time reversal of a stationary Markov chain is again a stationary Markov chain.

10. *Exact sampling.* Let S_0 be a finite set, and F be a random mapping from S_0 to S_0. For $x, y \in S_0$ put $P(x,y) := \mathbf{P}[F(x) = y]$. Moreover, let $F_1, F_2, \dots$ be a sequence of independent random mappings from S_0 to S_0, all having the same distribution as F. Show

a) If Y is an S_0-valued random variable independent of F, whose distribution is an equilibrium distribution of P, then $F(Y)$ has the same distribution as Y.

b) Assume that the random subsets R_n of S_0 defined by

$$R_n := F_1(F_2(\dots(F_n(S_0))\dots), \quad n = 1, 2, \dots$$

have an intersection $R := \bigcap R_n$ which $\mathbf{P}$-a.s. consists of only one element, i.e. $R = \{Z\}$ $\mathbf{P}$-a.s. for an S_0- valued random variable Z. Then P has exactly one equilibrium distribution, and this is the distribution of Z.
(Hint: Consider the sequence $Y_n := F_1(F_2(\dots(F_n(Y))\dots)$, $n = 1, 2, \dots$, where Y is as in a)).

11. Consider a simple random walk ($p = 1/2$) with absorbing boundaries on $\{0, 1, 2, \dots 10\}$. Suppose the following payoff function is given

$$[0, 2, 4, 3, 10, 0, 6, 4, 3, 3, 0].$$

Find the optimal stopping rule and give the expected payoff starting at each site.

12. (From: G.F. Lawler, Introduction to Stochastic Processes, Chapman & Hall, 1995)
Consider a simple "Wheel of Fortune" game. A wheel is divided into 12 equal-sized wedges. Eleven of the wedges are marked with the numbers $100, 200, \dots, 1100$ denoting an amount of money won if the wheel lands on those numbers. The twelfth wedge is marked "bankrupt". A player can spin as many times as he or she wants. Each time the wheel lands on a numbered wedge, the player receives that much money which is added to his/her previous winnings. However, if the wheel ever lands on the "bankrupt" wedge, the player loses all of his/her money that has been won up to that point. The player may quit at any time, and take all the money he or she has won (assuming the "bankrupt" wedge has not come up). Assuming that the goal is to maximize one's expected winnings in this game, devise an optimal strategy for playing this game and compute one's expected winnings as well as the probability of going bankrupt.

12P. (Free section, or in German: Kür)

a) Try a computer simulation to answer the questions in exercise 12. How many simulation runs does one need to determine the probability of going bankrupt (when using the optimal strategy) with an accuracy of approximately 1 % ? Estimate the expected number of steps till stopping, given one does not go bankrupt.

b) Somebody gives you a joker which puts you into the situation that the first *bankrupt* does not count. Assume you still decide to use the stopping strategy computed in exercise 12. How much does the expected winning increase compared to the situation without joker?