Übungen zur Vorlesung „ Stochastic Processes “

Abgabetermin: Dienstag, 14.05.02, in der Vorlesung

13. *Time reversal of the residual lifetime process.* Consider a renewal chain with life time distribution ϱ on $\mathbb{N}$ and finite expected life time $\mathbf{E}R$. Let ν be the equilibrium distribution of the residual lifetime process (Y_n), and $P(x,y)$ be its transition matrix. (Recall that $\nu(k) = \frac{1}{\mathbf{E}R}\mathbf{P}[R \geq k]$.)

a) Compute the "dual transition matrix" Q in the sense of Exercise 9a).

b) Compute the *hazard function*: the conditional probability that the lifetime R equals $k+1$, given that it exceeds k.

Congratulations! You have just seen that the time reversal of the staionary residual lifetime process is the stationary age process.

14. We consider three lifetime distributions:

$$\begin{aligned} \varrho_1(dr) &:= \frac{1}{10} 1_{[0,10]}(r)dr \\ \varrho_2(dr) &:= \delta_{10}(dr) \\ \varrho_3(dr) &:= 1_{[1,\infty)}(r)\frac{2}{r^3}dr \end{aligned}$$

a) For all three cases, plot the equilibrium density of the residual lifetime process. In which cases does convergence to equilibrium hold ?

b) Compute for all three cases

i) the expected lifetime,

ii) the expected residual lifetime in equilibrium,

iii) the expected size-biased lifetime.

15. ***The Gamma Distribution***

The Gamma(k)-distribution has density:

$$g_k(y) = \frac{1}{\Gamma(k)} y^{k-1} e^{-y}, \quad y > 0$$

(Recall that $\Gamma(k) = \int_0^\infty x^{k-1} e^{-x} dx$)

a) *Laplace transform:*

Let Y be Gamma(k)-distributed, $k > 0$. Show that

$$\mathbf{E}\, e^{-\beta Y} = (1+\beta)^{-k}, \quad \beta > 0.$$

(Hint: Substitute $(1+\beta)y$.)

b) Let $Y_i \quad (i = 1, 2)$ be Gamma(k_i)-distributed and independent. What is the distribution of $Y_1 + Y_2$?

(Hint: The distribution of an $\mathbb{R}_+$-valued random variable is uniquely determined by its Laplace transform.)

16. *Queuing*

In a single server bank customers arrive at time points forming a homogenous Poisson process with rate α. When the server is not free, the customers queue up. Let G be the distribution of the service times (assumed i. i d.)

a) Compute the expected number of customers arriving while one customer is served.

b) Assume that initially there are no customers waiting. Let Z_n be the number of people in the queue immediately after the n-th customer has been served. Then during the first busy period, i.e. until he server is free again, Z_n has the form $Z_n = 1 + X_1 + \ldots + X_n$, where the X_i are i.i.d. random variables. What is X_i? Assume that the service rate $\frac{1}{\mu}$ equals the arrival rate α. Can it happen that the line becomes so long that the server is never free again ?

For answering the last question, the following fact is helpful which we state without proof: An integer-valued random walk $X_n = x + Z_1 + Z_2 + \ldots + Z_n$ (where the Z_i are i.i.d. $\mathbb{Z}$-valued random variables) is recurrent provided the Z_i have expectation zero.

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Consider the situation of Exercise 16, assuming $\alpha = 1/\mu$ and G is an exponential distribution. Starting from one customer initially, guess if the system converges to equilibrium. What could be the expected time till the server will be free again?

Make a simulation study in which $\alpha = \frac{1}{\mu} = 1$. Keep track of the length of the line! What happens if $\alpha = 1$ and $\mu = 0.5$ or $\mu = 2$, respectively?