Übungen zur Vorlesung „Stochastic Processes“

Abgabetermin: Dienstag, 21.05.02, in der Vorlesung

17. (From: S. Ross, Stochastic processes, Wiley 1996, p. 154)
Consider a single-server bank in which potential customers arrive at a Poisson rate α but only enter when the server is free. Let G denote the distribution of the service time.

a) At what rate do customers enter the bank?
b) What fraction of potential customers enter the bank?
c) What fraction of time is the server busy?

18. ***Poisson decimated***
Let N be Poisson(α)-distributed and let $X_1, X_2, \ldots$ be a coin-tossing sequence with success probability p, independent of N. Let $S_n := \sum_{i=1}^{n} X_i$.

a) Compute $\mathbf{E} \exp(-\beta S_n), \quad \beta > 0$

b) Compute $\mathbf{E} \exp(-\beta S_N), \quad \beta > 0.$

(Hint: $\mathbf{E} f(S_N) = \sum_{n=0}^{\infty} P(N = n) E f(S_n)$)

c) So how is S_N distributed? (Recall the Laplace transform of the Poisson distribution which was computed in the course).

d) How could you have obtained the result of c) without computation by thinning a Poisson process ?

19. a) Let S_0 be a finite state space, P be a stochastic matrix and π be a probability distribution on S_0.
Show that π is an equilibrium distribution for P provided it fulfills the detailed balance equation

$$\pi(x)P(x,y) = \pi(y)P(y,x) \qquad (x, y \in S_0) \tag{1}$$

(i.e. if for the $(P;\pi)$-chain the flow from x to y is the same as the flow from y to x).Equilibrium distributions with the property (1) are called *reversible*.

b) Find an example of a transition probability with a non-reversible equilibrium distribution.

20. *This exercise is inspired by an example in O. Haeggstroem's nice lecture notes "Finite Markov chains and algorithmic applications" (available at* `http://www.math.chalmers.se/~olleh/surveys.html` *)*

Consider the $k \times k$ -grid $G = \{1, \ldots, k\} \times \{1, \ldots, k\}$ (consisting of k^2 "vertices" v), and say that two vertices $v, w \in G$ are *neighboured* if they have distance 1. Call $\{0,1\}^G$ the set of *configurations*. Say a vertex v is *occupied by* the configuration x if $x(v) = 1$. (Think of particle configurations on G described by mappings x from G to $\{0,1\}$ where $x(v) = 1$ if and only if the vertex v is occupied by a particle.)

Call a configuration x *allowed* (or *hard core*) if for *no* pair of neighboured sites v, w is jointly occupied by x. Put $S_0 :=$ the set of allowed configurations. Our objective is to find a transition probability P on S_0 which leaves the uniform distribution π on S_0 invariant. To this end we first come up with a "reference dynamics":

(R) Pick a vertex v at random, toss a fair coin, put $X_{n+1}(v) = 0$ if the coin comes up heads and $X_{n+1}(v) = 1$ if the coin comes up tails (and leave all the other $X_n(w)$, $w \neq v$, unchanged).

Obviously this can't be the solution (why?). But what about a modification:

(M) Same as (R), but in any case put $X_{n+1}(v) = 0$ if some neighbour w of v is occupied by X_n.

Show that (M) does the required job, rendering a transition probability P on S_0 which leaves the uniform distribution π on S_0 invariant. (Hint: Use Exercise 19.)

(By the way: P defined by (M) is even irreducible and aperiodic. Do you see an argument for this as well?)

20 P. Write a program for picking an (at least approximately) uniformly distributed hard-core configuration on G (in the set-up of Exercise 20). Use this to estimate the mean number of "particles" in a uniformly distributed hard-core configuration.