Übungen zur Vorlesung „ Stochastic Processes “

Abgabetermin: Dienstag, 29.05.02, in der Vorlesung

21. a) Let ν be a measure on $\mathbb{R}\setminus\{0\}$ obeying

$$\int (y^2 \wedge 1)\,\nu(dy) < \infty, \tag{1}$$

and Φ be a PPP on $\mathbb{R}\setminus\{0\}$ with intensity measure ν.

We saw in Lemma 3.5.3 that

$$S_\varepsilon := \int_{\{|y|>\varepsilon\}} y\,\Phi(dy) - \int_{\{\varepsilon\le|y|\le 1\}} y\,\nu(dy) =: I_\varepsilon + A_\varepsilon$$

converges as $\varepsilon \to 0$ (in probability) to a random variable S.

Discuss the "compensation term" A_ε and its limit as $\varepsilon \to 0$ in the following three examples:

(i) $\nu(dy) := y^{-2}dy$,
(ii) $\nu(dy) := y^{-2}1_{\mathbb{R}_+}(y)\,dy$,
(iii) $\nu(dy) := y^{-3/2}1_{\mathbb{R}_+}(y)\,dy$.

b) Consider

$$\nu(dy) := \text{ const } y^{-(1+\alpha)}dy.$$

Figure out for which $\alpha \in \mathbb{R}$ the measure ν meets condition (1).

(For such an α, and $S := \lim_{\varepsilon\to 0}\int_{\{|y|>\varepsilon\}} y\,\Phi(dy)$, $\mathcal{L}(S)$ is called a *symmetric α-stable distribution* on $\mathbb{R}$, and a Lévy Process (X_t) with $\mathcal{L}(X_1) = \mathcal{L}(S)$ is called a *symmetric α-stable process* on $\mathbb{R}$.)

22. (from S. Ross, Stochastic Processes, 2nd ed,. Wiley 1996)

A two-dimensional Poisson process is a random point configuration Φ in the plane such that (i) for any region B of area A, the number of points in B is Poisson distributed with mean λA, and (ii) the numbers of points in nonoverlapping regions are independent. Consider a fixed point $x \in \mathbb{R}^2$, and let X denote the distance from x to its nearest point in Φ, where distance is measured in the usual Euclidean manner. Show that:

a) $\mathbf{P}\{X > t\} = e^{-\lambda\pi t^2}$,
b) $\mathbf{E}[X] = 1/(2\sqrt{\lambda})$.

Let R_i, $i \ge 1$ denote the distance from an arbitrary point x to the ith closest point in Φ to it. Show that, with $R_0 = 0$,

c) $\pi R_i^2 - \pi R_{i-1}^2$, $i \ge 1$ are independent exponential random variables, each with rate λ.

23. a) Let π be a probability distribution on S_0, and R be a stochastic matrix on S_0. We want to modify R into a P such that the detailed balance equation holds:

$$\pi(x)R(x,y) = \pi(y)R(y,x), \quad x, y \in S_0. \tag{2}$$

To this purpose, let us fix $x \in S_0$ and let Z have distribution $R(x,.)$.
Now modify Z as follows. If $Z = y$, and $\pi(x)R(x,y) \le \pi(y)R(y,x)$, then accept the proposed outcome y, in putting

$$Y := y.$$

If $\pi(x)R(x,y) > \pi(y)R(y,x)$, then accept y only with probability

$$p := \frac{\pi(y)R(y,x)}{\pi(x)R(x,y)},$$

i.e. put

$$Y = \begin{cases} y & \text{with probabilty } p, \\ x & \text{with probabilty } 1-p. \end{cases}$$

Let $P(x,.)$ be the distribution of Y. Show that π and P indeed satisfy (2).

b) (Hastings algorithm) Let R be an irreducible stochastic matrix on S_0, and π be a probability distribution on S_0 with strictly positive weights whose ratios $r(x,y) := \frac{\pi(x)}{\pi(y)}$ are known for all x, y with $R(y,x) > 0$. Specify the dynamics of an irreducible Markov chain (X_n) with equilibrium distribution π in terms of R and r.

24. How does Exercise 20 fit into the framework of 23?