Übungen zur Vorlesung „Stochastic Processes"

Abgabetermin: Dienstag, 11.06.02, in der Vorlesung

29. a) Consider a population evolving in continuous time in which each individual independently of all the others (and of the prehistory) gives birth to a new individual at rate β and dies at rate δ. Argue that the Q-matrix of the population size process satisfies

$$Q(n,n+1) = n\beta, \quad Q(n,n-1) = n\delta, \quad Q(n,n) = -n(\beta+\delta).$$

b) Establish the backward equation for

$$u(t,n) := \mathbf{P}_n[X_t = 0].$$

From this, obtain a differential equation for

$$\phi(t) := \mathbf{P}_1[X_t = 0]$$

and compute $\phi(t)$. (Hint: Use that $\mathbf{P}_2[X_t = 0] = \mathbf{P}_1[X_t = 0]^2$).

30. Let $X_1, X_2, \ldots$ be i.i.d. integrable random variables, and

$$S_n := X_1 + \ldots + X_n.$$

Compute

a) $\mathbf{E}[X_1 | X_1 + X_2]$

b) $\mathbf{E}[X_1 | (S_n, S_{n+1}, \ldots)]$

c) $\mathbf{E}[\frac{X_1}{X_1+X_2}]$.

31. a) What is the conditional expectation of a random variable Z given a constant? (Note that also constants can be viewed as random variables!)

b) A transition probability $\Pi(x,dz)$ from S to $\mathbb{R}$ is called *conditional distribution* of Z given $X = x$ if the joint distribution of X and Z is of the form

$$\mathbf{P}[(X,Z) \in (dx,\, dz)] = \mathbf{P}[X \in dx]\,\Pi(x,dz).$$

Write $\mathbf{E}[Z|X]$ in terms of Π.

32. a) Recall that the correlation of two random variables Y_1 and Y_2 with positive variances σ_1^2 and σ_2^2 is given by

$$\kappa = \frac{\mathrm{Cov}[Y_1, Y_2]}{\sigma_1 \sigma_2}.$$

Show that $\frac{\sigma_2 \kappa}{\sigma_1} Y_1$ and $Y_2 - \frac{\sigma_2\kappa}{\sigma_1} Y_1$ are uncorrelated (i.e. have correlation zero). (Hint: You may assume without loss of generality (why?) that Y_1 and Y_2 have expectation 0.)

b) Let Z_1 and Z_2 be real-valued random variables with a joint normal distribution, expectations μ_1 and μ_2, variances σ_1^2 and σ_2^2, and correlation κ. Using the fact that random variables with a joint normal distribution are independent iff they are uncorrelated, show that the conditonal distribution of Z_2 given $Z_1 = z_1$ is normal with mean $\mu_2 + \frac{\sigma_2\kappa}{\sigma_1}(z_1 - \mu_1)$ and variance $\sigma_2^2(1-\kappa^2)$.