

## Übungen zur Vorlesung „Stochastic Processes“

Abgabetermin: Dienstag, 25.06.02, in der Vorlesung

**37.** Let  $(Z_k)$  be a coin tossing sequence with parameter  $p$ ,  $S_n := \sum_{k=1}^n Z_k$ .

For  $a \in \mathbb{Z}$  let  $T_a := \inf\{n \geq 0 | X_n = a\}$ . Let  $a < 0$ ,  $b > 0$ , and  $T := \min(T_a, T_b)$ .

- i) Assume  $p = \frac{1}{2}$ . Show that  $M_n := S_n^2 - n$ ,  $n = 0, 1, \dots$ , is a martingale, and use this to conclude that  $ET = |a| \cdot b$ .
- ii) Assume  $p < \frac{1}{2}$ ,  $q := 1 - p$ . Determine a constant  $c$  such that the process  $e^{cS_n}$ ,  $n = 0, 1, \dots$  is a martingale, and show that

$$P[T = T_a] = \frac{1 - \left(\frac{p}{q}\right)^b}{1 - \left(\frac{p}{q}\right)^{|a|+b}}.$$

**38.** Consider the following parameters:  $0 < d < u < \infty$ ;  $r \geq 0$ ,  $p \in ]0, 1[$ ,  $s_0 \in \mathbb{R}$ ,  $N \in \mathbb{N}$ . Assume that  $S_0, S_1, \dots, S_N$  is a sequence of real-valued random variables on a probability space  $(\mathcal{A}, \mathbf{P})$  with the following properties:

$S_0 = s_0$  a.s.; given  $(S_0, \dots, S_{n-1})$ , the random variable  $S_n$  takes the value  $uS_{n-1}$  with probability  $p$  and the value  $dS_{n-1}$  with probability  $1 - p$ .

- a) Find a necessary and sufficient condition for the parameters  $u, d, r$  and  $p$  such that there exists a probability measure  $\mathbf{P}^*$  which is a “strictly positive reweighting” of  $\mathbf{P}$  such that  $X_n := S_n(1+r)^{-n}$ ,  $n = 0, \dots, N$ , is an  $((\mathcal{F}_n), \mathbf{P})$ -martingale (with  $\mathcal{F}_n := \mathcal{A}(S_0, \dots, S_n)$ ,  $n = 0, \dots, N$ ).
- b) Let  $\mathbf{P}^*$  be a probability measure with the properties required in a). Why is every  $((\mathcal{F}_n), \mathbf{P}^*)$ -martingale  $(M_n)$  of the form

$$M_n = \sum_{k=1}^n \xi_k (X_k - X_{k-1})$$

with some  $(\mathcal{F}_n)$ -predictable  $(\xi_n)$  ?

**39. a)** The so called *reflection principle* says that for a standard Wiener process  $W$  and  $a > 0$ ,

$$\mathbf{P}\left[\sup_{0 \leq s \leq 1} W_s > a\right] = 2\mathbf{P}[W_1 > a.]$$

Use this and the Borel Cantelli-lemma to show that for all  $\varepsilon > 0$ ,

$$\mathbf{P}\left[\sup_{0 \leq s \leq 1} |W_{n+s} - W_n| > n^\varepsilon \text{ for infinitely many } n \in \mathbb{N}\right] = 0.$$

- b) Use the strong law of large numbers and a) to infer that

$$W_t/t \rightarrow 0 \text{ a.s. as } t \rightarrow \infty.$$

(By the way, concerning an asymptotic upper bound of  $|W_t|$  as  $t \rightarrow \infty$ , a much finer result is available. The so called *law of the iterated logarithm* states that

$$\limsup_{t \rightarrow \infty} W_t / \sqrt{2t \log \log t} = - \liminf_{t \rightarrow \infty} W_t / \sqrt{2t \log \log t} = 1.)$$

c) Let  $(W_t)$  be a standard Wiener process,  $\sigma > 0$ .

$$X_t := \sigma^{-\frac{1}{2}} W_{\sigma t} \quad (t \geq 0), \quad Y_t := t W_{1/t} \quad (t > 0), \quad Y_0 := 0.$$

Are  $(X_t)$  and  $(Y_t)$  standard Wiener processes as well?

40. Show that a standard Wiener process  $(W_t)$  in the time interval  $[0, 1]$  can be decomposed into

$$W_t = B_t + tW_1$$

with independent parts  $W_1$  and  $(B_t)$ .

- a) Characterize the finite dimensional distributions of  $(B_t)_{0 \leq t \leq 1}$ .
- b) Compute the conditional distribution of the pair  $(W_1, W_2)$ , given  $W_3 = x$  ( $x \in \mathbb{R}$ ).