Übungen zur Vorlesung „Stochastic Processes“

Abgabetermin: Dienstag, 25.06.02, in der Vorlesung

37. Let (Z_k) be a coin tossing sequence with parameter p, $S_n := \sum_{k=1}^{n} Z_k$.

For $a \in \mathbb{Z}$ let $T_a := \inf\{n \geq 0 | X_n = a\}$. Let $a < 0$, $b > 0$, and $T := \min(T_a, T_b)$.

i) Assume $p = \frac{1}{2}$. Show that $M_n := S_n^2 - n, n = 0, 1, \dots,$ is a martingale, and use this to conclude that $ET = |a| \cdot b$.

ii) Assume $p < \frac{1}{2}$, $q := 1 - p$. Determine a constant c such that the process e^{cS_n}, $n = 0, 1, \dots$ is a martingale, and show that

$$P[T = T_a] = \frac{1 - (\frac{p}{q})^b}{1 - (\frac{p}{q})^{|a|+b}}\,.$$

38. Consider the following parameters: $0 < d < u < \infty$; $r \geq 0$, $p \in]0, 1[$, $s_0 \in \mathbb{R}$, $N \in \mathbb{N}$. Assume that $S_0, S_1, \dots, S_N$ is a sequence of real-valued random variables on a probability space $(\mathcal{A}, \mathbf{P})$ with the following properties:

$S_0 = s_0$ a.s. ; given $(S_0, \dots, S_{n-1})$, the random variable S_n takes the value uS_{n-1} with probability p and the value dS_{n-1} with probability $1 - p$.

a) Find a necessary and sufficient condition for the parameters u, d, r and p such that there exists a probability measure $\mathbf{P}^*$ which is a “strictly positive reweighting” of $\mathbf{P}$ such that $X_n := S_n(1 + r)^{-n}$, $n = 0, \dots, N$, is an $((\mathcal{F}_n), \mathbf{P})$-martingale (with $\mathcal{F}_n := \mathcal{A}(S_0, \dots, S_n), n = 0, \dots, N$.)

b) Let $\mathbf{P}^*$ be a probability measure with the properties required in a). Why is every $((\mathcal{F}_n), \mathbf{P}^*)$- martingale (M_n) of the form

$$M_n = \sum_{k=1}^{n} \xi_k (X_k - X_{k-1})$$

with some $(\mathcal{F}_n)$-predictable (ξ_n) ?

39. a) The so called *reflection principle* says that for a standard Wiener process W and $a > 0$,

$$\mathbf{P}[\sup_{0 \leq s \leq 1} W_s > a] = 2\mathbf{P}[W_1 > a.]$$

Use this and the Borel Cantelli-lemma to show that for all $\varepsilon > 0$,

$$\mathbf{P}[\sup_{0 \leq s \leq 1} |W_{n+s} - W_n| > n^{\varepsilon} \text{ for infinitely many } n \in \mathbb{N}] = 0.$$

b) Use the strong law of large numbers and a) to infer that

$$W_t / t \to 0 \text{ a.s. as } t \to \infty.$$

(By the way, concerning an asymptotic upper bound of $|W_t|$ as $t \to \infty$, a much finer result is available. The so called *law of the iterated logarithm* states that

$$\limsup_{t\to\infty} W_t/\sqrt{2t\log\log t} = -\liminf_{t\to\infty} W_t/\sqrt{2t\log\log t} = 1.)$$

c) Let (W_t) be a standard Wiener process, $\sigma > 0$.

$$X_t := \sigma^{-\frac{1}{2}} W_{\sigma t} \quad (t \geq 0), \quad Y_t := tW_{1/t} \quad (t > 0), \quad Y_0 := 0.$$

Are (X_t) and (Y_t) standard Wiener processes as well?

40. Show that a standard Wiener processs (W_t) in the time interval $[0, 1]$ can be decomposed into

$$W_t = B_t + tW_1$$

with independent parts W_1 and (B_t).

a) Characterize the finite dimensional distributions of $(B_t)_{0\leq t\leq 1}$.

b) Compute the conditional distribution of the pair (W_1, W_2), given $W_3 = x \quad (x \in \mathbb{R})$.